SQ10 $a^2 - b^2 = (a+b)(a-b)$; $a^2 + b^2 = (a+ib)(a-ib)$

requires new math!

- Can $a^2 + b^2 + c^2$ be written as $(\text{something})^2$?

Call it $p_x^2 + p_y^2 + p_z^2$ (no loss of generality)

[can't do it by simple math, including "i"]

Regard (p_x, p_y, p_z) as components of a vector $\vec{p} = p_x \hat{i} + p_y \hat{j} + p_z \hat{k}$

Consider $\vec{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k}$

Pauli Matrices

$$\vec{\sigma} \cdot \vec{p} = p_x \sigma_x + p_y \sigma_y + p_z \sigma_z$$ [dot product of two vectors]

$$= p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} p_z & p_x + i p_y \\ p_x - i p_y & -p_z \end{pmatrix}$$

$$\underbrace{(\vec{\sigma}\cdot\vec{p})^2}_{\substack{\text{2x2 matrices}\\ \text{as components}}} = \begin{pmatrix} p_z & p_x+ip_y \\ p_x-ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x+ip_y \\ p_x-ip_y & -p_z \end{pmatrix} = \begin{pmatrix} p_x^2+p_y^2+p_z^2 & 0 \\ 0 & p_x^2+p_y^2+p_z^2 \end{pmatrix}$$

$\therefore$ Want to write $p_x^2+p_y^2+p_z^2$ as [something]^2, need to expand math thoughts to 2x2 matrices!

So what?

$$E^2 = \underbrace{m^2c^4}_{\text{consider simple case of massless particle}} + c^2(p_x^2+p_y^2+p_z^2)$$

$$E^2 - c^2(p_x^2+p_y^2+p_z^2) = 0$$

$$\Rightarrow E^2 - c^2\underbrace{(\vec{\sigma}\cdot\vec{p})^2}_{\text{2x2 matrix}} = 0 \Rightarrow \left(E + \underbrace{c(\vec{\sigma}\cdot\vec{p})}_{\text{2x2 matrix}}\right)\left(E - c\underbrace{(\vec{\sigma}\cdot\vec{p})}_{\text{2x2 matrix}}\right) = 0$$

Done!

So what? Implication? Relativistic QM equation

Recall: Schrödinger Equation
for free particle [non-relativistic]

$$E = \frac{p^2}{2m} \quad (E\text{-}p \text{ relation})$$

$$E\psi = \frac{p^2}{2m}\psi \quad (\text{quantum})$$

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi \quad (\text{Schrödinger})$$

Relativistic (but massless) Quantum Equation?

$$(E + c(\vec{\sigma}\cdot\vec{p}))\underbrace{(E - c(\vec{\sigma}\cdot\vec{p}))\psi} = 0$$

or simply $(E - c(\vec{\sigma}\cdot\vec{p}))\psi = 0$

(then substitute $E \to i\hbar\frac{\partial}{\partial t}$; $\hat{p}_x \to \frac{\hbar}{i}\frac{\partial}{\partial x}$, etc)

Done!

This is the quantum equation
for massless fermions!

spin-$\frac{1}{2}$

More explicitly, it is a 2x2 equation

$$\left[E\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - c\begin{pmatrix}p_z & p_x+ip_y\\ p_x-ip_y & p_z\end{pmatrix}\right]\underbrace{\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}}_{} = 0$$

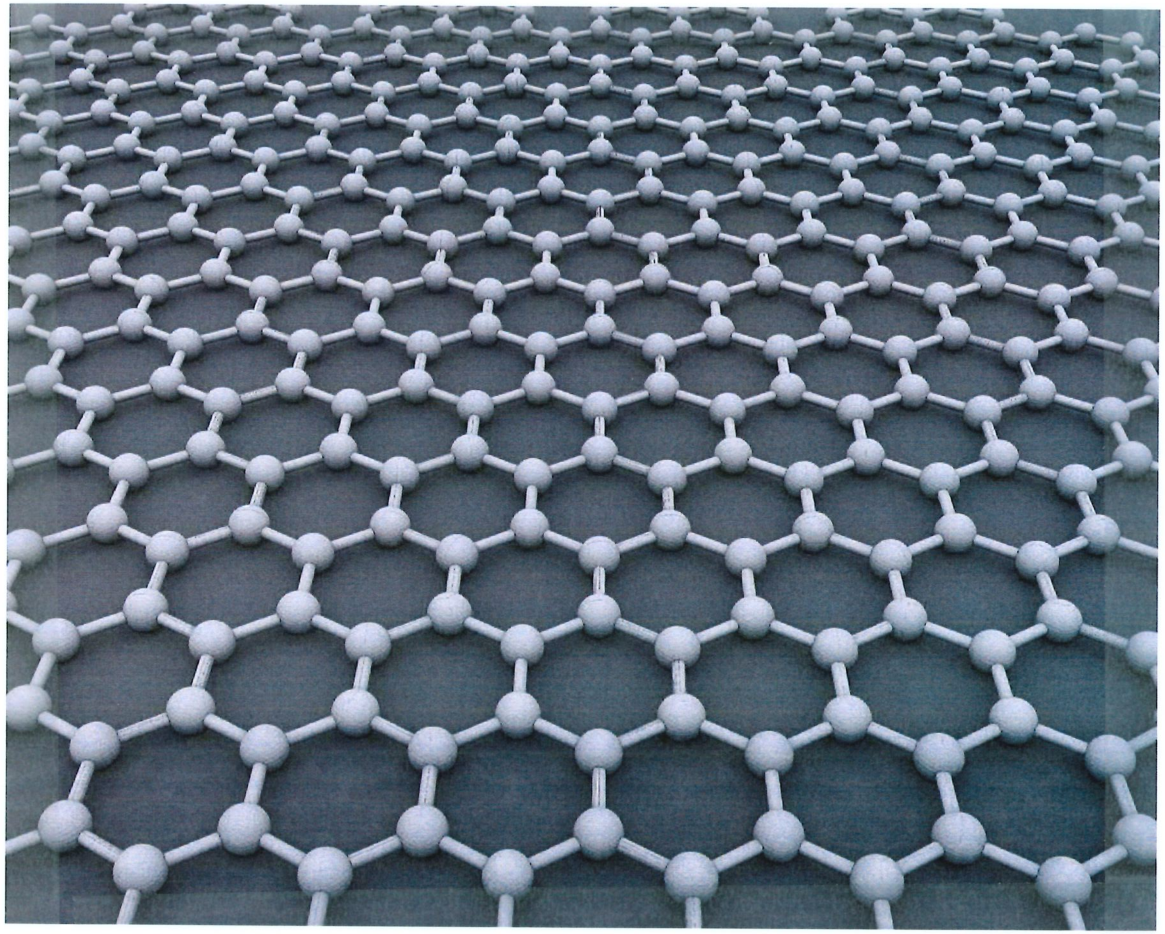
Note: The wavefunction consists of $\underbrace{\text{two functions stacking up}}_{\text{imply spin-}\frac{1}{2}}$

$$\begin{pmatrix}E-cp_z & -c(p_x+ip_y)\\ -c(p_x-ip_y) & E-cp_z\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0 = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left.\begin{aligned}(E-cp_z)\psi_1 - c(p_x+ip_y)\psi_2 &= 0\\ (E-cp_z)\psi_2 - c(p_x-ip_y)\psi_1 &= 0\end{aligned}\right\}\text{ 2 equations!}$$

then introduce operators, do separation of variables (e.g. $\psi_1(\vec{r},t) = \psi_1(\vec{r})T(t)$), etc.

Relevance: Graphene (1 layer of carbon atoms) [2010 Nobel Prize]



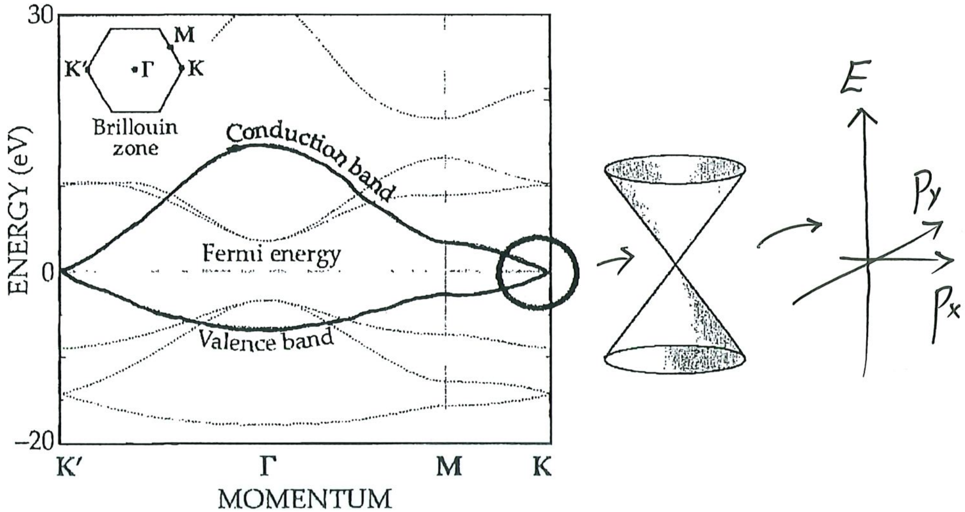
2D system

one sheet from graphite

Electrons move in 2D system (p_x, p_y)

How does energy E depend on (p_x, p_y)?

Geim and MacDonald, Physics Today (August 2007), p. 35



$$E \sim p \rightsquigarrow \sqrt{p_x^2 + p_y^2}$$

$$\hat{H} = v_F \begin{pmatrix} 0 & p_x - i p_y \\ p_x + i p_y & 0 \end{pmatrix}$$

some speed
(Fermi velocity)

$$= v_F \, \vec{\sigma} \cdot \vec{p}$$

Table Top system with behaviour of massless fermion!

Further Extension

How about writing $E^2 - m^2c^4 - c^2(p_x^2 + p_y^2 + p_z^2) = 0$

into $[E + (\text{something})][E - (\text{something})] = 0$?

Another Nobel Prize [Dirac 1933]

$$\underbrace{[E - (\text{something})]}_{\text{involve } \underline{4\times4 \text{ matrices}}} \psi = 0$$ is <u>Dirac Equation</u>

$$\begin{pmatrix} \times \\ \times \\ \times \\ \times \end{pmatrix} \begin{array}{l} \} \text{ spin up/down } (+\text{ve } E) \\ \} \text{ spin up/down } (-\text{ve } E) \end{array}$$

column